Lecture 6:

Recall:

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k, l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k, l)\, e^{-j 2\pi \left(\frac{km}{M} + \frac{ln}{N} \right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j \sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p, q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m, n)\, e^{j 2\pi \left(\frac{pm}{M} + \frac{qn}{N} \right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (\mathcal{U}_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$.

Theorem: $\mathcal{U}^* \mathcal{U} = \frac{1}{N} I$ where $\mathcal{U}^* = (\overline{\mathcal{U}})^T$ (conjugate transpose)

$$\mathcal{U} \mathcal{U}^* = \frac{1}{N} I.$$

$$\therefore \mathcal{U}^{-1} = (N\mathcal{U})^*$$

$(\overline{a+jb} = a - jb)$

$(\overline{e^{j\theta}} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
$= e^{-j\theta})$

Image decomposition by DFT

Suppose $\hat{g} = DFT(g) = \mathcal{U} g \mathcal{U}$

Then: $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^* \mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^T$ ← Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

Remark:

Note that $UU^* = \frac{1}{N} I$. $\therefore$ U is not unitary.

If we normalize U to $\tilde{U} = \sqrt{N}\, U$. Then $\tilde{U}$ is unitary!

Some other definition of DFT:

(1D) $\hat{f}(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{k-1} f(k) e^{-j\left(\frac{2\pi m k}{N}\right)}$

(2D) $\hat{f}(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l)\, e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$

In this case, let $\tilde{U} = (\tilde{U}_{kl})_{0 \le k,l \le N-1}$; $\tilde{U}_{kl} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi k l}{N}}$. Then: $\hat{f} = \tilde{U} f \tilde{U}$

Then, $\tilde{U} = \sqrt{N}\, U$

$\therefore$ Normalizing the definition of DFT $\Rightarrow$ unitary $\tilde{U}$ can be applied!

BUT: Inverse DFT must be adjusted!!

Mathematics of JPEG (Optional)

Consider a $M \times N$ image f. Extend f to a $2M \times 2N$ image $\tilde{f}$, whose indices are taken from $[-M, M-1]$ and $[-N, N-1]$.

Define $f(k,l)$ for $-M \le k \le M-1$ and $-N \le l \le N-1$ such that

$$f(-k-1,-l-1) = f(k,l) \quad \} \text{ Reflection about } (-1/2,-1/2)$$

$$\left.\begin{aligned} f(-k-1,l) &= f(k,l) \\ f(k,-l-1) &= f(k,l) \end{aligned}\right\} \text{ Reflection about the axis } k=-1/2 \text{ and } l=-1/2$$

Example:

$l = -1/2$

$$\begin{array}{cccccc|l} 9 & 8 & 7 & 7 & 8 & 9 & k=-3 \\ 6 & 5 & 4 & 4 & 5 & 6 & k=-2 \\ 3 & 2 & 1 & 1 & 2 & 3 & k=-1 \\ 3 & 2 & 1 & 1 & 2 & 3 & k=0 \\ 6 & 5 & 4 & 4 & 5 & 6 & k=1 \\ 9 & 8 & 7 & 7 & 8 & 9 & k=2 \\ l=-3 & l=-2 & l=-1 & l=0 & l=1 & l=2 & \end{array}$$

$k = -1/2$

$f(-2,-2)$

$=$

$f(1,1)$

Reflection about $(-1/2,-1/2)$.

$f(-1,1)$

$=$

$f(0,1)$

Reflection about the axis $k=-1/2$.

Make the extension as a reflection about $(0, 0)$, the axis $k=0$ and the axis $\ell=0$.

Done by shifting the image by $(\frac{1}{2}, \frac{1}{2})$

After shifting

$$
\begin{array}{cccccc|l}
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+(-3) \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+(-2) \\
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+(-1) \\
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+0 \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+1 \\
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+2 \\
\hline
\frac{1}{2}+-3 & \frac{1}{2}+-2 & \frac{1}{2}+-1 & \frac{1}{2}+0 & \frac{1}{2}+1 & \frac{1}{2}+2 &
\end{array}
$$

k

ℓ

Now, we compute the DFT of (shifted) $\tilde{f}$:

$$F(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j\frac{2\pi}{2M}m(k+\frac{1}{2})} e^{-j\frac{2\pi}{2N}n(l+\frac{1}{2})}$$

$$= \frac{1}{4MN} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

$$= \frac{1}{4MN}(\underbrace{\sum_{k=-M}^{-1} \sum_{l=-N}^{-1}}_{A_1} + \underbrace{\sum_{k=-M}^{-1} \sum_{l=0}^{N-1}}_{A_2} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=-N}^{-1}}_{A_3} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=0}^{N-1}}_{A_4})$$

$$f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

After some messy simplication, we can get:

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

Definition: (Even symmetric discrete cosine transform [EDCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M-1$ and $0 \leq l \leq N-1$. The **even symmetric discrete cosine transform (EDCT)** of f is given by:

$$\hat{f}_{ec}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

with $0 \leq m \leq M-1, 0 \leq n \leq N-1$

Remark:
- Smart idea to get a decomposition consisting only of cosine function (by reflection and shifting!)
- Can be formulated in matrix form
- Again, it is a separable image transformation.

- The inverse of EDCT can be explicitly computed. More specifically, the **inverse EDCT** is defined as:

$$f(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{ec}(m,n)\cos\frac{\pi m(2k+1)}{2M}\cos\frac{\pi n(2l+1)}{2N} \qquad (**)$$

where $C(0)=1, C(m)=C(n)=2$ for $m,n\neq 0$

Also involving cosine functions only!

- Formula (**) can be expressed as matrix multiplication:

$$f = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}_{ec}(m,n)\vec{T}_m \vec{T'}_n^T$$

elementary images under EDCT!

where: $\vec{T}_m = \begin{pmatrix} T_m(0) \\ T_m(1) \\ \vdots \\ T_m(M-1) \end{pmatrix}, \vec{T'}_n = \begin{pmatrix} T'_n(0) \\ T'_n(1) \\ \vdots \\ T'_n(N-1) \end{pmatrix}$ with $T_m(k) = C(m)\cos\frac{\pi m(2k+1)}{2M}$

and $T'_n(k) = C(n)\cos\frac{\pi n(2k+1)}{2N}$.

This is what JPEG does!!

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $$g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$$

$\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$

Then, the DFT of $g * w$ $= MN\, DFT(g)\, DFT(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Proof:

Change of variables:

$n \to n'' = n - n'$

$m \to m'' = m - m'$

DFT of $g*w$ at (p, q)

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m')\, w(n', m')\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \underbrace{\frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n', m')\, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)}}_{\hat{\omega}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n, m)$ and $g(n, m-M) = g(n, m)$

$$\therefore\ T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j2\pi\frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j 2\pi \frac{pn''}{N}} \overset{n'''=N+n''}{=\!=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N,\, m'')}_{g(n''',\, m'')}\, e^{-j 2\pi \left(\frac{pn''}{N}\right)} \cancelto{1}{e^{j2\pi p}}$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'', m'')\, e^{-j 2\pi \left(\frac{pn''}{N} + \frac{qm''}{M}\right)} = MN\, \hat{g}(p, q)$$

$$\therefore\ \widehat{g * w}\,(p, q) = MN\, \hat{g}(p, q)\, \hat{w}(p, q)$$

Note:

(Spatial domain) $I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\Big\downarrow \ \text{DFT}$$

(Frequency domain) $MN \, \hat{I} \underbrace{\odot}_{\text{pixel-wise multiplication}} \hat{g}$ (Modifying the Fourier coefficients by multiplication)

2. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$$-k_0 \le k' \le N-1-k_0 \quad \text{and} \quad -l_0 \le l' \le N-1-l_0$$

Let $\tilde{g}$ be shifted image of g defined as:

$$\tilde{g}(k, l) = g(k-k_0, l-l_0) \quad \text{where} \quad 0 \le k \le N-1, \quad 0 \le l \le N-1$$

Then:

$$\hat{\tilde{g}}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k-k_0, l-l_0)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\frac{1}{N^2} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l'=-l_0}^{N-1-l_0} g(k', l')\, e^{-j2\pi\left(\frac{k'm+l'n}{N}\right)}}_{\hat{g}(m,n)} e^{-j2\pi\left(\frac{k_0m+l_0n}{N}\right)}$$

$$\therefore \quad \hat{\tilde{g}}(m,n) = \hat{g}(m,n)\, e^{-j2\pi\left(\frac{k_0m+l_0n}{N}\right)}$$