

Lecture 6:

Recall:

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k, l))_{k, l}$, where $0 \leq k \leq M-1$, $0 \leq l \leq N-1$ is defined as:

$$\hat{g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k, l) e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p, q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m, n) e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

(no $\frac{1}{Mn}$!) DFT of g (no -ve sign)

DFT in Matrix form

Theorem: Consider a $N \times N$ image g , the DFT of g can be written as:

$$\hat{g} = U g U \quad (\text{DFT in matrix form})$$

where $U = (U_{kl})_{0 \leq k, l \leq N-1} \in M_{N \times N}$ and $U_{kl} = \frac{1}{N} e^{-j \frac{2\pi k l}{N}}$.

Theorem: $U^* U = \frac{1}{N} I$ where $U^* = (\overline{U})^T$ (conjugate transpose)

$$U U^* = \frac{1}{N} I.$$

$$\therefore U^{-1} = (N U)^*$$

$$(\overline{a + jb} = a - jb)$$

$$(\overline{e^{j\theta}} = \cos\theta + j \sin\theta = \cos\theta - j \sin\theta = e^{-j\theta})$$

Image decomposition by DFT

$$\text{Suppose } \hat{g} = \text{DFT}(g) = U g U$$

$$\text{Then: } U U^* = \frac{1}{N} I = U^* U$$

$$\therefore g = (N U)^* \hat{g} (N U)$$

$$\therefore g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^T \leftarrow \text{Elementary image of DFT}$$

$$\text{where } \vec{w}_k = k^{\text{th}} \text{ col of } (N U)^*$$

Remark:

Note that $UU^* = \frac{1}{N}I$. $\therefore U$ is not unitary.

If we normalize U to $\tilde{U} = \sqrt{N}U$. Then \tilde{U} is unitary!

Some other definition of DFT:

$$(1D) \quad \hat{f}(m) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f(k) e^{-j\left(\frac{2\pi mk}{N}\right)}$$

$$(2D) \quad \hat{f}(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$$

In this case, let $\tilde{U} = (\tilde{U}_{kl})_{0 \leq k, l \leq N-1}$; $\tilde{U}_{kl} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi kl}{N}}$. Then:

$$\text{Then, } \tilde{U} = \sqrt{N}U$$

$$\hat{f} = \tilde{U} f \tilde{U}$$

\therefore Normalizing the definition of DFT \Rightarrow unitary \tilde{U} can be applied!

BUT: Inverse DFT must be adjusted!!

Mathematics of JPEG (Optional)

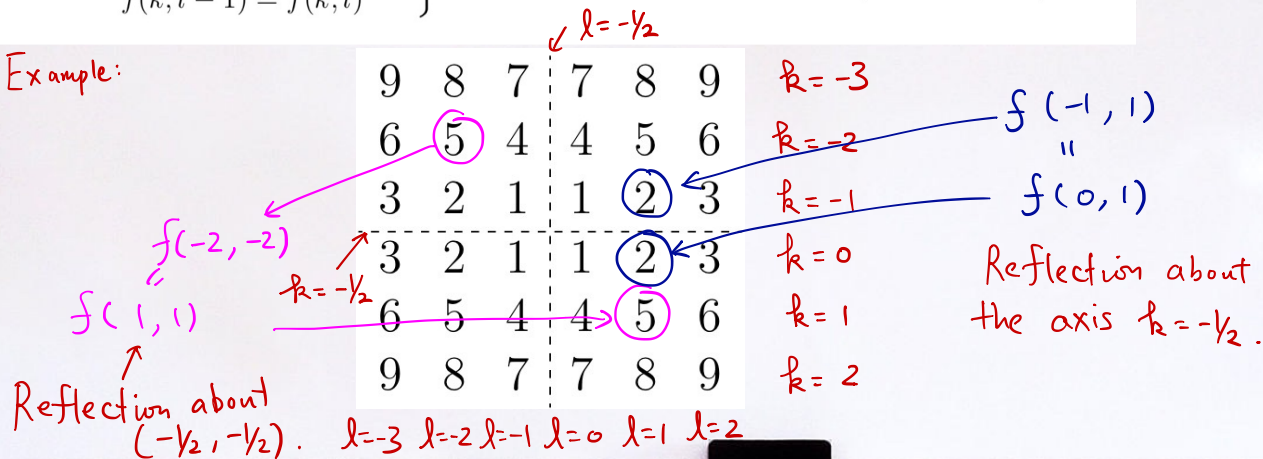
Consider a $M \times N$ image f . Extend f to a $2M \times 2N$ image \tilde{f} , whose indices are taken from $[-M, M-1]$ and $[-N, N-1]$.

Define $f(k, l)$ for $-M \leq k \leq M-1$ and $-N \leq l \leq N-1$ such that

$$f(-k-1, -l-1) = f(k, l) \quad \left. \vphantom{f(-k-1, -l-1)} \right\} \text{Reflection about } (-1/2, -1/2)$$

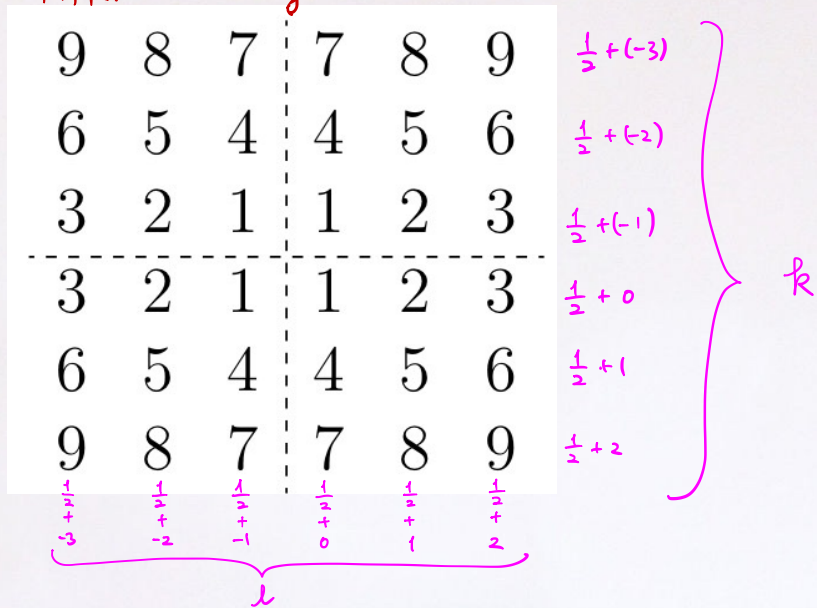
$$\left. \begin{aligned} f(-k-1, l) &= f(k, l) \\ f(k, -l-1) &= f(k, l) \end{aligned} \right\} \text{Reflection about the axis } k = -1/2 \text{ and } l = -1/2$$

Example:



Make the extension as a reflection about $(0, 0)$, the axis $k=0$ and the axis $l=0$.
 Done by shifting the image by $(\frac{1}{2}, \frac{1}{2})$

After shifting



Now, we compute the DFT of (shifted) \tilde{f} :

$$\begin{aligned}
 F(m, n) &= \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k, l) e^{-j\frac{2\pi}{2M}m(k+\frac{1}{2})} e^{-j\frac{2\pi}{2N}n(l+\frac{1}{2})} \\
 &= \frac{1}{4MN} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k, l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))} \\
 &= \frac{1}{4MN} \left(\underbrace{\sum_{k=-M}^{-1} \sum_{l=-N}^{-1}}_{A_1} + \underbrace{\sum_{k=-M}^{-1} \sum_{l=0}^{N-1}}_{A_2} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=-N}^{-1}}_{A_3} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=0}^{N-1}}_{A_4} \right) \\
 &\quad f(k, l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}
 \end{aligned}$$

After some messy simplification, we can get:

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l) \cos \left[\frac{m\pi}{M} \left(k + \frac{1}{2} \right) \right] \cos \left[\frac{n\pi}{N} \left(l + \frac{1}{2} \right) \right]$$

Definition: (Even symmetric discrete cosine transform [EDCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M - 1$ and $0 \leq l \leq N - 1$. The **even symmetric discrete cosine transform (EDCT)** of f is given by:

$$\hat{f}_{ec}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l) \cos \left[\frac{m\pi}{M} \left(k + \frac{1}{2} \right) \right] \cos \left[\frac{n\pi}{N} \left(l + \frac{1}{2} \right) \right]$$

with $0 \leq m \leq M - 1, 0 \leq n \leq N - 1$

- Remark:
- Smart idea to get a decomposition consisting only of cosine function (by reflection and shifting!)
 - Can be formulated in matrix form
 - Again, it is a separable image transformation.

- The inverse of EDCT can be explicitly computed. More specifically, the **inverse EDCT** is defined as:

$$f(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C(m)C(n) \hat{f}_{ec}(m, n) \cos \frac{\pi m(2k+1)}{2M} \cos \frac{\pi n(2l+1)}{2N} \quad (**)$$

where $C(0) = 1, C(m) = C(n) = 2$ for $m, n \neq 0$

Also involving cosine functions only!

- Formula (**) can be expressed as matrix multiplication:

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}_{ec}(m, n) \vec{T}_m \vec{T}_n^T$$

elementary images under EDCT!

where: $\vec{T}_m = \begin{pmatrix} T_m(0) \\ T_m(1) \\ \vdots \\ T_m(M-1) \end{pmatrix}, \vec{T}_n^T = \begin{pmatrix} T'_n(0) \\ T'_n(1) \\ \vdots \\ T'_n(N-1) \end{pmatrix}$ with $T_m(k) = C(m) \cos \frac{\pi m(2k+1)}{2M}$

and $T'_n(k) = C(n) \cos \frac{\pi n(2k+1)}{2N}$.

This is what JPEG does!!

Why is DFT useful in imaging:

1. DFT of convolution:

$$\text{Recall: } g * w(n, m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} g(n-n', m-m') w(n', m')$$

$$(g, m \in M_{N \times M}(\mathbb{R}))$$

Then, the DFT of $g * w = MN \text{ DFT}(g) \text{ DFT}(w)$

\therefore DFT of convolution can be reduced to simple multiplication!

Proof:

$$\text{DFT of } g * w \text{ at } (p, q)$$

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g * w(n, m) e^{-j2\pi(\frac{pn}{N} + \frac{qm}{M})}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m') w(n', m') e^{-j2\pi(\frac{pn}{N} + \frac{qm}{M})}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n', m') e^{-j2\pi(\frac{pn'}{N} + \frac{qm'}{M})} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'', m'') e^{-j2\pi(\frac{pn''}{N} + \frac{qm''}{M})}}_{T(p, q)}$$

$\hat{w}(p, q)$

$T(p, q)$

Change of variables:

$$n \rightarrow n'' = n - n'$$

$$m \rightarrow m'' = m - m'$$

Note that: g and w are periodically extended.

$$\therefore g(n-N, m) = g(n, m) \text{ and } g(n, m-M) = g(n, m)$$

$$\therefore T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'', m'') e^{-j2\pi \frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'', m'') e^{-j2\pi(\frac{pn''}{N})}$$

Consider $\sum_{n''=-n'}^{-1} g(n'', m'') e^{-j2\pi \frac{pn''}{N}} \stackrel{n''=N+n'}{=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N, m'')}_{g(n'', m'')} e^{-j2\pi (\frac{pn''}{N})} e^{j2\pi p}$

We can do similar thing for index m'' .

$$\therefore T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'', m'') e^{-j2\pi (\frac{pn''}{N} + \frac{qm''}{M})} = MN \hat{g}(p, q)$$

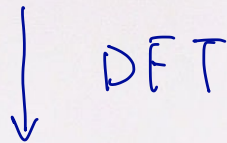
$$\therefore \widehat{g * w}(p, q) = MN \hat{g}(p, q) \hat{w}(p, q)$$

Note.

(Spatial domain)

$$I * g$$

(Linear filtering:
Linear combination of
neighborhood pixel
values)



(Frequency domain)

$$MN \hat{I} \odot \hat{g}$$

pixel-wise
multiplication

(Modifying the
Fourier coefficients
by multiplication)

2. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$$-k_0 \leq k' \leq N-1-k_0 \quad \text{and} \quad -l_0 \leq l' \leq N-1-l_0$$

Let \tilde{g} be shifted image of g defined as:

$$\tilde{g}(k, l) = g(k - k_0, l - l_0) \quad \text{where } 0 \leq k \leq N-1$$

$$\begin{aligned} \text{Then: } \hat{\tilde{g}}(m, n) &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k - k_0, l - l_0) e^{-j2\pi \left(\frac{km + ln}{N} \right)} \\ &= \frac{1}{N^2} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l'=-l_0}^{N-1-l_0} g(k', l') e^{-j2\pi \left(\frac{k'm + l'n}{N} \right)} e^{-j2\pi \left(\frac{-k_0 m + -l_0 n}{N} \right)} \\ &\quad \underbrace{\hspace{10em}}_{\hat{g}(m, n)} \end{aligned}$$

$$\therefore \hat{\tilde{g}}(m, n) = \hat{g}(m, n) e^{-j2\pi \left(\frac{-k_0 m + -l_0 n}{N} \right)}$$

